Optimal navigation in complex networks

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(Received 26 September 2008; revised manuscript received 22 January 2009; published 6 April 2009)

Recent literature has presented evidence that the study of navigation in complex networks is useful to understand their dynamics and topology. Two main approaches are usually considered: navigation of random walkers and navigation of directed walkers. Unlike these approaches ours supposes that a traveler walks optimally in order to minimize the cost of the walking. If this happens, two extreme regimes arise—one dominated by directed walkers and the other by random walkers. We try to characterize the critical point of the transition from one regime to the other in function of the connectivity and the size of the network. Furthermore, we show that this approach can be used to generalize several concepts presented in the literature concerning random navigation and direct navigation. Finally, we defend that investigating the extreme regimes dominated by random walkers and directed walkers is not sufficient to correctly assess the characteristics of navigation in complex networks.

DOI: 10.1103/PhysRevE.79.046103 PACS number(s): 89.75.Hc, 02.50.Ga, 89.65.−s

I. INTRODUCTION

Recent literature has presented evidence that the study of navigation is useful to understand the dynamics and topological properties of complex networks. In general, two main different approaches have been considered, namely, navigation of random walkers and directed walkers in complex networks. In both approaches, a walk consists of stepping from node to node of the complex network via the links between them.

In the first approach, at each time step, a random walker located at a specific node chooses one of the neighbors of this node based on some transition matrix in order to continue the walk. This approach has introduced several different concepts that were used to improve our understanding about complex networks. In [1], the mean first passage time is exactly derived for arbitrary networks and a quantity known as random walk centrality is introduced. In [2], random walk searches are considered and different possibilities of the transition matrix are studied. In [3] networks are sampled using different types of random walks and the properties of the sampled networks are compared to the original networks.

The second approach takes the opposite direction and considers that at each step the walker takes the shortest path to the target. This walker is like a traveler in a new city who, by asking questions about the way to his destination, arrives there using the minimal number of steps. In this context, for example, Sneppen *et al.* [4] defined the search information, which is the total information needed to identify one shortest path between a source and a target. In practice, what the search information measures is the number of yes/no questions necessary for the walker to know which direction to take in order to arrive closer to his target. Based on this, it was possible to introduce the idea of the node with minimal access information (the node which provides the best access to the entire network) and the best node to be hidden (the node which requires the maximum number of questions to be found). These ideas were used to study a large class of complex networks [4–6]. In [7], the idea provided by the search information was weighted according to the betweenness of the links to consider the notion that the large roads usually take you closer to the targets than small roads. Using these two measures, in [7] the authors could deal directly with the difference between targets located at small and large distances. In [8], the navigation with limited information is considered assuming that in each node the amount of information that a traveler has access is limited. In this case, the authors show that the walker usually travels distances substantially longer than the actual shortest path and the effect of not choosing the correct link in a node depends on the degree of the node.

In this paper, we consider that a traveler in a new city wants to go from a given source to a specific target. However, differently from the above mentioned papers, in each step of the walk, he can either ask other people which is the correct path to follow or randomly follow one of the paths available in this step. We assume that there are two constant costs associated to the trajectory, namely, a cost associated to the walk from one node to its neighbor and a cost associated to get information about the trajectory from other people. Thus, we suppose that the traveler will make the optimal decision in order to minimize the sum of the costs associated to the complete trajectory. If this happens, we show that two different regimes arise. In one extreme, when the cost of getting information is low when compared to the cost of walking a step, the decision is made in order to minimize the distance of the trajectory. Conversely, at the other extreme when the cost of getting information is high, the walk converges in the limit to a random walk. We try to characterize the transition from one regime to the other in function of the connectivity and the size of the network. Additionally, we show that the solution of this problem can be used to characterize complex networks and generalizes several measures introduced in the context of random and direct navigation.

It is worth mentioning that this is not the first time that a kind of optimization principle is used to understand the structure and dynamics of complex systems. In [9–12], for instance, it is shown that complex networks may arise from optimization principles. Furthermore, in [13,14] optimization has been used to study the complex human dynamics of task execution.

II. SETUP OF THE PROBLEM

Suppose that we have a city represented by a network G with n nodes $V(G)=\{1,2,\ldots,t,\ldots,n\}$, where t is a special node called the target. In each node, a traveler has to choose between making the next step of his walk randomly at a cost C_N or using the link that will certainly approximate him to the target t with cost C_N+C_I, where C_N is the constant cost of one step navigation and C_I is the constant cost of asking people for the correct direction. Furthermore, we assume that the traveler in each node of the network makes his decision in order to find

$$J(i) = \min_{\pi\in\Pi} E_\pi\left[\sum_{k\in\mathcal{P}(i,t)} g(k,u(k))/i\right], \tag{1}$$

where $\mathcal{P}(i,t)$ is the path from node i to the target t [15], the expectation $E^\pi[\cdot/i]$ is conditional on the policy π and node i, $\pi=\{u(1),u(2),\ldots,u(t),\ldots,u(n)\}$ is an admissible policy belonging to the set of admissible policies Π, and $u(k)$ is an admissible control belonging to the set of admissible controls $U(k)=\{0,1\}$, $\forall k$. In particular, here we assume that $u(k)=0$ if in node k he decides to randomly choose the next step of his walk and $u(k)=1$ if in node k he decides asking other people the correct path to follow, which implies that in this case the traveler will follow the correct one. The cost per stage $g[k,u(k)]$ is the cost of using $u(k)$ at node k given by

$$g[k,u(k)] = \begin{cases} C_N & \text{if } u(k)=0,\ \forall\, k\neq t \\ C_N+C_I & \text{if } u(k)=1,\ \forall\, k\neq t. \end{cases} \tag{2}$$

The intuition behind Eq. (2) is simply that if the traveler did not reach the target, he will have to pay at least more C_N. If he additionally decides to ask for the right path, he has to pay C_N+C_I.

It is clear that a traveler in a new city in practice will never exactly solve the problem presented in Eq. (1). In order to solve this problem, he has to know *a priori* the structure of the network of the city given here by $p_{kl}(u)$, $\forall u \in U(k)$, and $k,l\in V(G)$—the transition matrix from a node k to a node l if the control law u is used in node k. However, the solution of this problem deserves to be studied for at least three reasons: it is an upper bound of the quality of the behavior of the traveler, it is an interesting way to characterize navigation in complex networks, and it overlaps the random walk and the directed walk behaviors. Note that if the ratio C_I/C_N is high enough, the traveler will never choose to ask people the correct direction to follow. On the other hand, if the ratio C_I/C_N is low, it is always optimal to ask for the correct direction since it avoids the deviation from the shortest paths.

Since the set $U(k)$ is finite for all k and assuming that the network here has no isolated clusters, $p_{tt}(u)=1$ for all $u \in U$ and $g(t,u)=0$ for all $u\in U$, this problem can be formulated as a stochastic shortest path problem [16]. Therefore, it is easy to show that the solution of problem (1) is given by the Bellman equation [17],

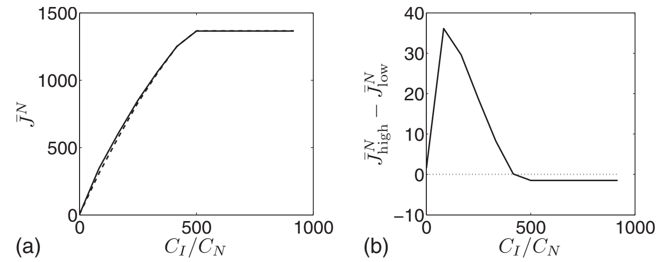

FIG. 1. (a) shows $\bar{J}^N_{\rm high}$ (solid line) and $\bar{J}^N_{\rm low}$ (dashed line) for several values of C_I/C_N. (b) shows $\bar{J}^N_{\rm high}-\bar{J}^N_{\rm low}$ for several values of C_I/C_N.

$$J(i) = \min_{u\in U(i)}\left[g(i,u)+\sum_{j=1}^{n} p_{ij}J(j)\right]. \tag{3}$$

Since by hypothesis $g(t,u)=0$ and $p(t,j)=0$, $\forall j\neq t$, then $J(t)=0$. Thus Eq. (3) says that $J(i)$, the optimal cost that the traveler being in node $i\in V(G)$ has to pay to reach node t, is divided in two parts: the cost that the traveler has to pay in node i plus the cost that the traveler has to pay in all the other nodes before reaching t. Define $\bar{J}^N(i)$, $\forall i\in V(G)$ as the average of $J(i)$ over all the targets t. If $\pi=\{1,1,\ldots,1\}$ then $\bar{J}^N(i)$ is known to be the characteristic path length of the node multiplied by the cost C_N+C_I. On the other hand, if $\pi=\{0,0,\ldots,0\}$, $\bar{J}^N(i)$ is the mean first passage time from i multiplied by C_N. Therefore, $\bar{J}^N(i)$ measures in average the difficulty to go from node $i\in V(G)$ to any another node $j\in V(G)-\{i\}$ if the traveler follows the optimal policy. This variable extends the idea of the so-called *minimal access information* defined in [4,6] for the case of directed walkers. Define also $\bar{J}^H(i)$, $\forall i\in V(G)$ as the average of all $J(j)$, $\forall j\in V(G)-\{i\}$, when $t=i$. This measures the ability that a node has to be hidden in a network generalizing the idea of the so-called *hide* defined in [4,6] for directed walkers. Finally, define $\bar{\bar{J}}$ as the average of J over all nodes and all targets. Hence, this variable measures the overall difficulty of navigating in the network.

In the rest of this paper, we analyze some networks using numerical solutions of the Bellman equation [Eq. (3)] found by means of the value iteration algorithm [16,18].

III. SCALE-FREE NETWORKS

We have applied these ideas to a typical Barabasi-Albert free-scale network [19] with 250 nodes. In order to have a better understanding of the results, we have divided the nodes of this network into two samples of 125 nodes according to the size of $\bar{J}^N$ when $C_I/C_N=0$. $\bar{J}^N_{\rm high}$ ($\bar{J}^N_{\rm low}$) is meant to be the average of $\bar{J}^N$ for the fraction of nodes that have highest (lowest) $\bar{J}^N$ when $C_I/C_N=0$. One should note that since this division is made when $C_I/C_N=0$, the nodes with lowest (highest) $\bar{J}^N$ are exactly the ones that have smallest (largest) characteristic path lengths.

While Fig. 1(a) shows the evolution of $\bar{J}^N_{\rm high}$ and $\bar{J}^N_{\rm low}$ for

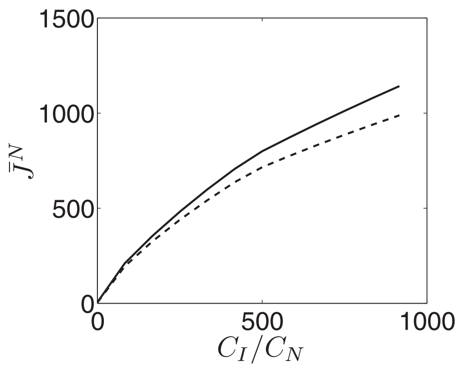

FIG. 2. This figure shows the average of the randomized versions of $\bar{J}^N_{\text{high}}$ (solid line) and $\bar{J}^N_{\text{low}}$ (dashed line) over ten realizations.

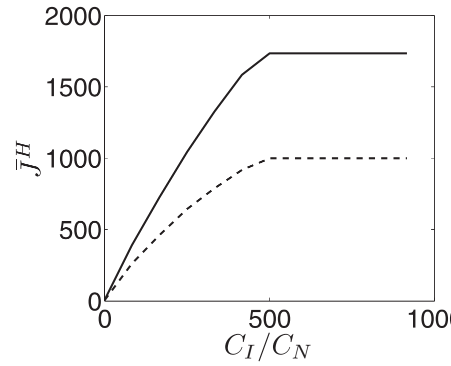

FIG. 3. This figure shows $\bar{J}^H_{\text{high}}$ (solid line) and $\bar{J}^H_{\text{low}}$ (dashed line) for several values of C_I/C_N.

several values of C_I/C_N, Fig. 1(b) shows the difference between these values $\bar{J}^N_{\text{high}}-\bar{J}^N_{\text{low}}$ for several values of C_I/C_N. One may identify two different regimes in this figure. The transition from the first regime to the second regime occurs when the optimal process of navigation is not dominated by directed walkers anymore but rather by random walkers. It is worth mentioning that the critical point C_I/C_N [located at the beginning of the plateau in Fig. 1(b)] is the ratio C_I/C_N where ceases the existence of the directed walk regime. Therefore, the process of navigation which takes place in complex networks interpolates the process of navigation driven by directed walkers which happens for low values of C_I/C_N and the process of navigation driven by random walkers which happens for high values of C_I/C_N.

The difficulty of navigation in a node $i \in V(G)$ depends strongly on the type of policy that is used. This is quite intuitive. While in the regime low C_I/C_N the level of difficulty of navigation in a node is driven by the shortest paths from this node to the others, in the regime high C_I/C_N the level of difficulty of navigation in a node is also influenced by the degree of this node. While $\bar{J}^N_{\text{low}}$ accounts for the difficulty of navigation from nodes with smallest characteristic path lengths, $\bar{J}^N_{\text{high}}$ accounts for the difficulty of navigation from the nodes that have largest characteristic path lengths. In general, there is a significant correlation between the nodes of the network that formed the sample used to evaluate $\bar{J}^N_{\text{high}}$ and the nodes of the network that have smallest degrees.

Note that in order to take the phenomenon presented in Fig. 1(b) into account one has to evaluate $\bar{J}^N$ for nodes in a typical network like we have done here. As shown in Fig. 2 the average of $\bar{J}^N$ over several randomized networks that conserves the degrees of the nodes and ensures the full connectivity of the network [20,21] does not present this phenomenon. This happens because the average of $\bar{J}^N(i)$ for a typical node i over the randomized networks in average depends only on the degree of node i. In each realization of the randomization process each node is located in a different place in the network breaking down (or reducing) the correlation between the degree of a node and its characteristic path length.

The transition between regimes of directed walkers and random walkers presented in Figs. 1(a) and 1(b) should arise in all types of networks where there is a positive correlation between the sample of nodes used to build $\bar{J}^N_{\text{high}}$ and the nodes with the smallest degrees of the network. For different networks, the important change in Fig. 1(b) is the value of the critical C_I/C_N, which depends on the topological characteristics of the network such as size and connectivity.

We have also built $\bar{J}^H_{\text{high}}$ and $\bar{J}^H_{\text{low}}$ using the same procedure presented above, but the choice of the two samples was based on the size of $\bar{J}^H$. Figure 3 shows the evolution of $\bar{J}^H_{\text{high}}$ and $\bar{J}^H_{\text{low}}$ for several values of C_I/C_N. The only phenomenon in this case is the increase in the ability of being hidden when the cost of navigation also increases.

A simple extension of the problem studied above, which follows the same lines in [4–6], is to consider that the cost provided by Eq. (2) is given by $C_I = c \times \text{degree}(k)$ where c is a constant, i.e., the cost of information is proportional to the degree of a node. The results of this extension also present the same pattern of Figs. 1(a) and 1(b) (not shown).

Figure 4 explores the difficulty of navigation in several scale-free networks. In particular, Fig. 4(a) compares the cost of navigation in a typical scale-free network with their maximally and minimally hierarchical versions built by the algorithm presented in [22] and ensuring the full connectivity of the network. Based on this figure, we have found that it is easier to navigate in the maximally hierarchical version than

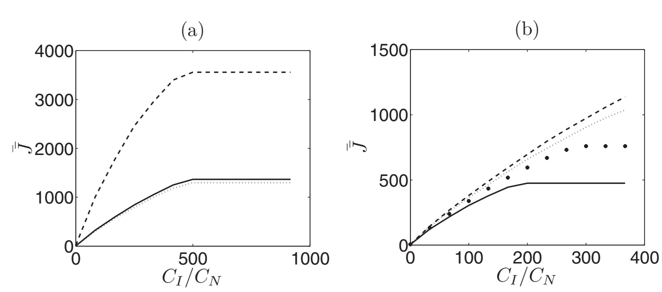

FIG. 4. (a) shows $\bar{\bar{J}}$ for several values of C_I/C_N for a typical scale-free network (solid line), for its maximally hierarchical version (dotted line), and for its minimally hierarchical version (dashed line). (b) shows $\bar{\bar{J}}$ for typical scale-free networks for several values of C_I/C_N and for different sizes: 100 (solid line), 150 (star), 200 (dotted line), and 250 (dashed line).

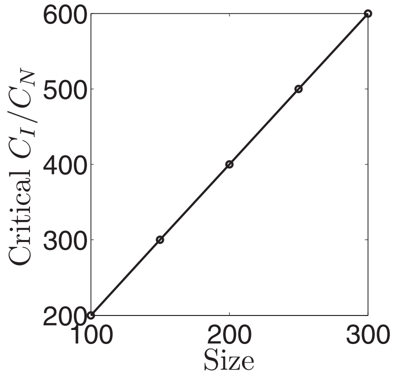

FIG. 5. The relation between the critical ratio C_I/C_N and the size for typical scale-free networks.

in the original scale-free network. Additionally, it is easier to navigate in the original scale-free network than in the minimally hierarchical network. On the other hand, Fig. 4(b) investigates the relation between the difficulty of navigation in a scale-free network and its size. As shown in this figure, it is more difficult to navigate in larger networks.

Figure 5 presents the relation between the critical ratio C_I/C_N and the size of the scale-free networks. This figure shows that we have found an almost straight line with positive slope coefficient between the critical ratio C_I/C_N and the size of the scale-free network. This happens due to the preferential attachment property present in scale-free networks. If the decision of the traveler is $u(i)=1$ in a given node i with a given degree in a neighborhood of the critical point, this happens because this node has degree in average larger than the rest of the network (since the ratio C_I/C_N is relatively high at this point). Furthermore, if new nodes are introduced in this network, it is likely that this node (with degree in average larger than the rest of the network) will be preferentially attached by some of these nodes. Therefore, the degree of this node will become larger and the traveler will accept to pay more for information in this node.

IV. RANDOM NETWORKS

As already stated, the pattern presented in Figs. 1(a) and 1(b) also arises in random networks (not shown).

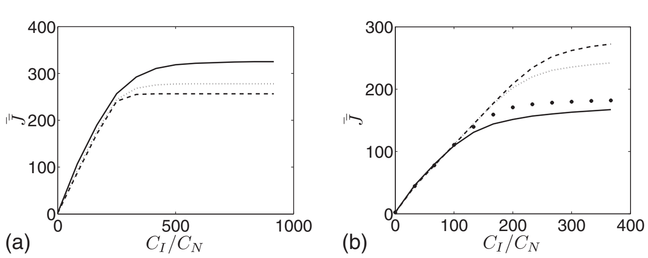

FIG. 6. (a) $\bar{\bar{J}}$ for typical random networks for several values of C_I/C_N with different probabilities of connection among the nodes: $p=0.05$ (solid line), $p=0.1$ (dotted line), and $p=0.2$ (dashed line). (b) $\bar{\bar{J}}$ for typical random networks for several values of C_I/C_N with different sizes: 100 (solid line), 150 (star), 200 (dotted line), and 250 (dashed line).

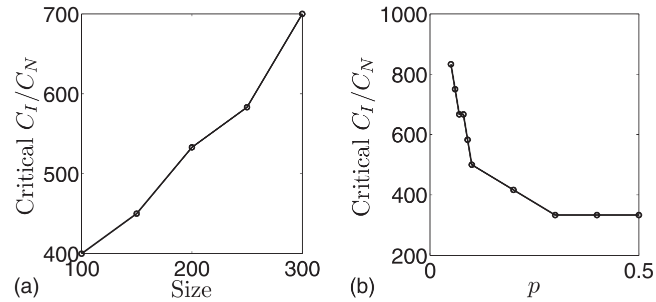

FIG. 7. (a) shows the relation between the critical ratio C_I/C_N and the size of typical random networks. (b) shows the relation between the critical ratio C_I/C_N and the connectivity of typical random networks.

Figure 6 evaluates the difficulty of navigation in several random networks. While Fig. 6(a) shows that more connected random networks have lower costs of navigation, Fig. 6(b) reinforces the results of Fig. 4(b) showing that it is more costly to navigate in larger networks.

We have also characterized the value of the critical ratio C_I/C_N for representative random networks. While Fig. 7(a) presents the results of the investigation of the relation between the critical ratio and the size of a random network, Fig. 7(b) explores the relation between the critical ratio and the connectivity of a random network. Figure 7(a) reinforces the results of Fig. 5. However, since the preferential attachment property is not present in random networks, the relation between the critical ratio C_I/C_N and the size of the network is not a straight line as the relation presented in Fig. 5. Figure 7(b) shows that the critical ratio C_I/C_N decreases with the increasing of the probability p of connection of typical random networks until the stabilization of the critical ratio C_I/C_N around a given value since now the network is almost all connected. This happens because when the network becomes more connected the paths become shorter. In fact, in more connected networks, if the traveler follows the wrong path in a given node, he can correct his mistake in a closer node.

V. REAL NETWORKS

We have also applied this methodology in order to study the navigation in some real networks, namely, (a) the Boston

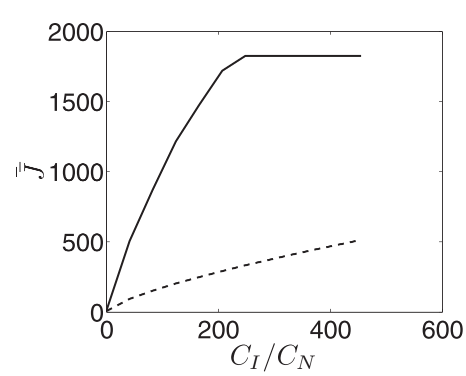

FIG. 8. This figure compares the difficulty of navigation $\bar{\bar{J}}$ in the Boston underground transportation system (solid line) with the average of its randomized version over ten different realizations (dashed line).

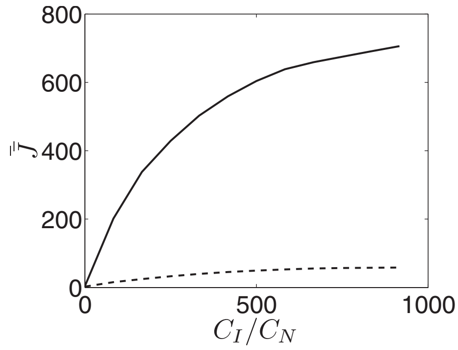

FIG. 9. This figure compares the difficulty of navigation $\bar{\bar{J}}$ in a representative subgraph with 250 nodes of the Swedish city Umeå mapped into an information network (solid line) with the average of its randomized version over ten different realizations (dashed line).

underground transportation system [23], (b) representative subgraphs [24] with 250 nodes of the Swedish city Umeå mapped into an information network [5], (c) representative subgraphs with 250 nodes of the Swedish city Malmö mapped into an information network [5], (d) representative subgraphs with 250 nodes of the Swedish city Stockholm mapped into an information network [5], (e) the US airlines connections network [25], and (f) the Zachary Karate club social network of friendships at a US university in 1970 [26].

Figures 8–13 compare the navigation in each network with its randomized version [20,21]. In general these figures show that in the original networks it is more difficult to navigate than their random counterparts. The same idea is shown in [5]. Some exceptions are the subgraphs of the information network of Stockholm. Therefore, in this case, we could not find just one representative subgraph. Thus, we have presented two representative subgraphs in Fig. 11. As one can see in this figure, while in one of the representative subgraphs it is more difficult to navigate than in the random counterpart network, in the other it is easier to navigate. The explanation for this phenomenon is the high heterogeneity of this network and the presence of real "islands" in the city [5]. Depending on the part of the city (represented here by a subgraph), the subgraphs of information network of this city have totally different properties. In particular, this result emphasizes the importance of investigating the accuracy of the properties of the whole network inferred from the available sampled information as considered in [3].

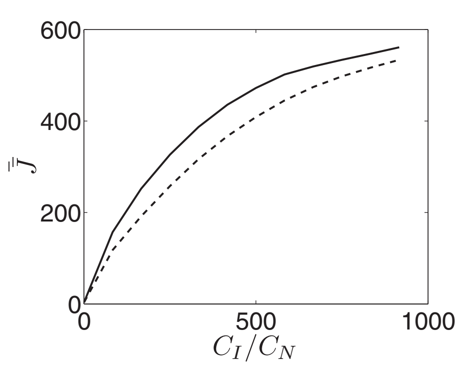

FIG. 10. This figure compares the difficulty of navigation $\bar{\bar{J}}$ in a representative subgraph with 250 nodes of the Swedish city Malmö mapped into an information network (solid line) with the average of its randomized version over ten different realizations (dashed line).

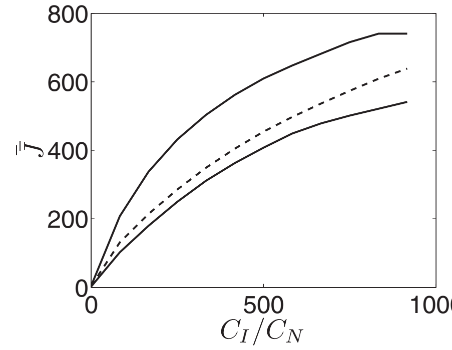

FIG. 11. This figure compares the difficulty of navigation $\bar{\bar{J}}$ in two representative subgraphs with 250 nodes of the Swedish city Stockholm mapped into an information network (solid line) with the average of its randomized version over ten different realizations (dashed line).

Figures 8–13 also reinforces the idea that in order to quantify the difficulty of navigation in complex networks it is not enough to look into one of the extremes (small C_I/C_N or large C_I/C_N) or even both extremes. For instance, Fig. 12 for small values of C_I/C_N reinforces the results in [5] showing that the cost of navigation in the US airline network is close to the random counterpart. However, the difference increases with the size of C_I/C_N.

There is a great difference between the difficulty of navigation in the original networks of the Boston underground transportation system and the city of Umeå and their randomized counterparts in Figs. 8 and 9. This is likely to be due to the real (geometric) constraints that the real networks are subjected to as suggested in [5]. In fact, the only constraint that the randomized networks are subject to is the degree distribution of the original network. In particular, in the case of the Boston underground transportation system (Fig. 8), as pointed out in [23], this transportation system was built with a very low cost in the sense that only a small number of edges was used. Furthermore, in real life, it is expected that only navigation in the directed walk regime occurs in this system, which according to [23] can be estab-

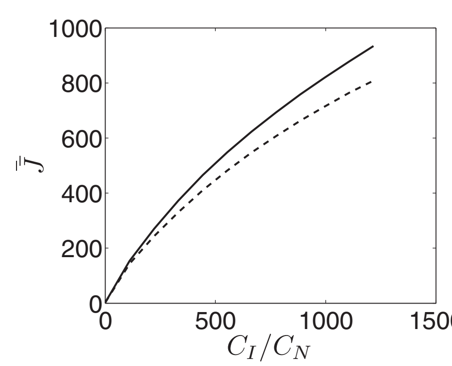

FIG. 12. This figure compares the difficulty of navigation $\bar{\bar{J}}$ in the US airlines connection network (solid line) with the average of its randomized version over ten different realizations (dashed line).

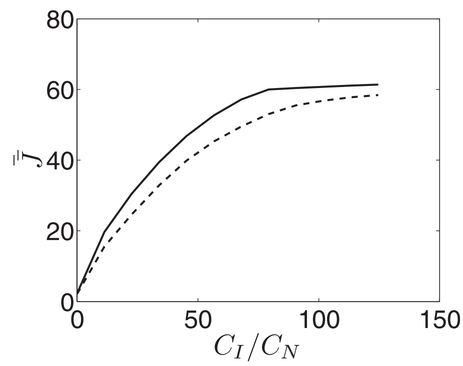

FIG. 13. This figure compares the difficulty of navigation $\bar{\bar{J}}$ in the Zachary karate club social network of friendships at a US university in 1970 (solid line) with the average of its randomized version over ten different realizations (dashed line).

lished with a satisfactory level of efficiency. As one may note in Fig. 8, even with strong geometrical constraints that this transportation system is subjected, the difference of difficulty of navigation between the original network and its randomized counterpart in the directed walk regime is not so large.

In this section, for comparison purposes, we also considered the issue of analyzing the difficulty of navigation in the Zachary karate club social network. At first sight it can be considered odd to analyze navigation in social networks, but really, it is not. For instance, in the directed walk regime, $\bar{J}^H(i)$ in social networks may be interpreted as the difficulty of accessing a member i of the network. On the other hand, $\bar{J}^H(i)$, in social networks in the random walk regime, may be interpreted as the difficulty in finding a member i of the network who performs a given role. The problem is that the person who is looking for this member does not know who this member is or where this member is hierarchically located. In the intermediate regimes, the interpretation is a combination of these ideas. $\bar{J}^N(i)$ may be interpreted analogously.

VI. FINAL REMARKS

This paper has discussed a framework to study optimal navigation in complex networks. The main idea of this framework is that unlike the usual approaches such as navigation of random walkers and navigation of directed walkers, we assumed that a traveler walks optimally in order to minimize the cost of walking. Furthermore, the cost of walking was divided in the so-called cost of navigation C_N and cost of information C_I. Based on these very simple premises, we showed that in the limits low C_I/C_N and high C_I/C_N the solution of this problem converges to the directed walker regime and the random walker regime, respectively.

The solution of the optimal navigation problem was achieved through the numerical solution of the Bellman equation associated to this problem. This solution was used to characterize theoretical networks, real networks, and the critical point of the transition from the directed walk regime to the random walk regime. It was also used to generalize several concepts presented in the literature in the context of random navigation and direct navigation.

It is important to state that one limitation of the methodology proposed here is the so-called curse of dimensionality, i.e., the huge computational cost associated to the numerical solution of the Bellman equation [Eq. (3)] when the number of states (nodes) is large. Because of that, as one might observe, the networks or the subgraphs of the networks analyzed in this work have a small number of nodes. Some possible routes used to deal with this limitation are in general based on approximations of the cost function J [16,18,27]. Therefore, to propose how to efficiently approximate the cost function is an interesting path of research. These approximations would allow us, for instance, to analyze the navigation in the actual networks of information of the Swedish cities instead in the subgraphs as we have done here.

Although the framework presented in this paper can be easily adapted to be valid for weighted networks [28], for comparison purposes, we have considered all the networks in this paper undirected and unweighted.

ACKNOWLEDGMENTS

The author is grateful to the Brazilian agency CNPQ for financial support. The author also wishes to thank Professor Martin Rosvall (Department of Biology, University of Washington) and Vito Latora (Dipartimento di Fisica, Universita di Catania) who kindly provided the data of the networks of the Swedish cities and the data of the network of the Boston underground transportation system, respectively.

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[24] It is worth noting that the subgraphs used to explore navigation in the Swedish cities were built using the following procedure. Start with an empty subgraph. Add to this subgraph a randomly selected node of the network (we will call this node "1"). Then add one of the neighbors of node 1 (we will call this neighbor node "2") to the subgraph. If the desired number of nodes of the subgraph is not reached, add another neighbor of node 1 and so on. If the desired number of nodes of the subgraph is not reached, after all the neighbors of node 1 are included in the subgraph, repeat the procedure to node 2 and so on. After choosing the desired number of nodes, connect these nodes with the original edges.

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[28] The only modification in the framework needed to deal with weighted networks is to replace the cost C_N in Eq. (2) by the size of the edge.